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Transient Response of Rayleigh Beams Transporting Moving Distributed Masses on Pasternak Foundation: Rayleigh-Ritz and Runge-Kutta Techniques

Adebola Samuel Adeoye[1], Tolulope Olamide Adeloye[2], Yahya Abdul Oluborode[3]

[1]Department of Mathematical Sciences, Achievers University, Owo, Ondo State, Nigeria

[2]Department of Mathematical Sciences, Nigeria Maritime University, Warri, Delta State, Nigeria

[3]Federal Medical Center, Idi Aba, Ogun State, Nigeria

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Abstract: **The dynamic response of structural elements to moving loads is a fundamental problem in structural dynamics with direct relevance to civil engineering, mechanical systems, and biomechanics. This study presents a computational framework for capturing the transient behavior of Rayleigh beams' models that incorporate rotary inertia and shear deformation subjected to moving distributed masses while supported by a Pasternak (two-parameter) elastic foundation. The Pasternak model, which combines Winkler-type vertical stiffness with shear-layer interaction, increases the mathematical complexity of the governing equations. Spatial discretization is achieved via the Rayleigh–Ritz method by representing transverse deflection as a linear combination of admissible functions that satisfy clamped boundary conditions; this reduces the governing fourth-order partial differential equation to a system of second-order ordinary differential equations in time. Temporal integration is performed with the classical fourth-order Runge–Kutta (RK4) scheme to resolve the time-varying effects of the moving mass and to balance accuracy with computational efficiency. Parametric studies investigate the influence of foundation stiffness, rotary inertia, flexural rigidity and moving-mass characteristics on deflection, natural frequencies and transient amplification. Results presented as distinct, non-overlapping plots demonstrate the sensitivity of dynamic response to individual parameters and confirm the proposed method's efficacy and adaptability for advanced modelling of engineering and biomechanical systems under dynamic loading.**

Keywords: **Structural dynamics, Rayleigh -Ritz, Runge-Kutta, Foundation stiffness, Rotary inertia, Flexural rigidity.**

I. INTRODUCTION

The dynamic response of beams under moving loads is a central challenge in modern engineering, with significant implications for high-speed transportation systems, advanced structural health monitoring, and biomechanical applications. Traditional beam theories, such as Euler-Bernoulli and Timoshenko formulations, have provided valuable insights; however, they often fall short when complex factors including rotary inertia, shear deformation, and variable foundation properties become significant. In response, the Rayleigh beam model has emerged as a superior alternative, offering improved accuracy in capturing the effects of both rotary inertia and shear deformation. In practical applications, beams are frequently subjected to moving distributed masses, such as vehicles crossing bridges or dynamic loads experienced by human bones during locomotion. When these beams rest on elastic supports modeled by Pasternak foundations, which account for both Winkler stiffness and shear layer interactions, the governing partial differential equations (PDEs) become highly complex. To manage this complexity, our research adopts a two-pronged computational approach. First, the spatial domain is discretized using the Rayleigh-Ritz method. In this method, the beam's deflection is approximated by a linear combination of admissible shape functions that satisfy the essential boundary conditions. This procedure reduces the original PDE to a system of coupled second-order ordinary differential equations (ODEs), effectively capturing the spatial variations in a finite-dimensional setting. Despite this reduction, accurately integrating the resulting temporal dynamics remains a formidable task, particularly under the influence of time-dependent moving loads. To address this, a classical fourth-order Runge-Kutta method is employed as the time integration solver. The Runge-Kutta method offers a robust and accurate mechanism for advancing the state of the system, ensuring that the transient responses critical for understanding resonance phenomena and stability are captured with high precision while maintaining computational efficiency.

Early work by Euler and Bernoulli laid the foundation for beam theory, yet these classical models neglect critical factors such as rotary inertia and shear deformation. This led to the development of more refined approaches like the Timoshenko beam theory Timoshenko, (Timoshenko 1921) which partially addressed these deficiencies by incorporating shear effects. Nevertheless, as the complexity of dynamic loading scenarios such as those involving moving distributed masses became apparent, the limitations of these formulations also surfaced. In recent decades, the Rayleigh beam model has emerged as a robust alternative, capturing the important influence of rotary inertia without the full complexity of Timoshenko's theory. This model has proven particularly effective in scenarios where high-speed loading conditions and slender structural elements are present. However, when coupled with sophisticated support conditions, such as those represented by the Pasternak foundation, (Pasternak 1954), further complexity arises. The Pasternak model, which supplements the Winkler foundation, (Winkler 1867) by introducing shear layer effects, provides a more realistic representation of soil-structure interaction and non-uniform elastic supports. Ogunlusi et al. (2024) analyzed non-uniform Rayleigh beams under accelerating moving masses on a bi-parametric foundation, applying Galerkin-based discretization and a fourth-order Runge–Kutta solver for transient response. They dosvovered that increased shear stiffness and rotary inertia reduce deflection amplitudes, and moving mass cases reach resonance at lower speeds compared to moving forces. Hammed et al. (2020) examined simply-supported beams on Pasternak foundations under distributed moving loads, using analytical methods. They found that higher foundation stiffness (K_o) and shear parameter (G_o) reduce deflection amplitudes, and higher velocities amplify dynamic responses. Studies by (Vlasov 1967) and subsequent researchers have emphasized the need to consider these advanced foundation models to accurately predict the dynamic response of structures. The interplay between a moving distributed mass and the structural dynamics of beams and plates has been explored extensively in the literature. Early investigations into moving loads, Jeffcott (1929) provided insights into the fundamental behaviors.

However, subsequent research revealed that the dynamic amplification and resonance phenomena arising from such loads necessitate a more detailed analytical treatment. For instance, Esmailzadeh and Ghorashi (1997) demonstrated that neglecting inertial effects of the moving mass can lead to significant under predictions of response amplitudes. More recent advances by Fryba (1999) have applied finite element methods (FEM) to capture these transient behaviors, yet these approaches are computationally intensive, especially for real-time or multi-parameter studies. A pivotal advancement in computational structural dynamics has been the adoption of the Rayleigh-Ritz method for spatial discretization. By approximating the displacement field as a series expansion of admissible shape functions which inherently satisfy the essential boundary conditions, this method reduces the governing partial differential equations to a system of coupled second-order ordinary differential equations (ODEs). This reduction not only simplifies the problem, but also allows for a more manageable analysis of complex geometries and material inhomogeneities. The efficiency and versatility of the Rayleigh-Ritz method have been demonstrated in numerous applications, from bridge dynamics Nguyen and Tran (2015) to biomechanical modeling of human bones (Zhang et al 2020). Kanwal et. al, 2024 compared vibrational behavior of various beam theories (Euler–Bernoulli, Timoshenko, Rayleigh) over Pasternak, Winkler, and Hetényi supports. Their modal analysis showed that shear deformation and rotary inertia raise natural frequencies significantly. Chen et al. (2024) developed a dynamic stiffness method for Levinson beams on Winkler–Pasternak foundations with axial loading. They offered closed-form dynamic-stiffness matrices and used mode-superposition to capture transient and buckling behaviour, showing axial load and foundation stiffness notably influence natural frequencies.

Ogunlusi et al. (2024) used Runge–Kutta temporal integration combined with Galerkin discretization to solve Rayleigh beam ODEs under variable-speed loads, providing validated amplitude and resonance predictions . Wang et al. (2023) improved Rayleigh–Ritz approximations using enhanced Fourier series, highlighting the method's utility in complex boundary problems. Li et al. (2023) modeled Timoshenko beams on fractional-order viscoelastic Pasternak foundations, analyzing free vibration and wave attenuation. They confirmed foundation damping greatly mitigates response amplitude. Bao and Liu (2020) evaluated Winkler vs Pasternak foundation models in soil-structure dynamics, finding Pasternak's shear layer crucial for accurate frequency prediction. Kapoor et al. (2023) introduced physics-informed neural networks to simulate both direct and inverse moving-load problems via a PINN framework, offering a modern alternative to classical numerical schemes. Adeoye and Awodola, (2018) investigated the response to moving distributed masses of pre-stressed uniform Rayleigh beams on variable elastic Pasternak foundation using generalized Galerkin techniques. The influence of rotatory inertial correction factor on the vibration of elastically supported non-uniform Rayleigh beam on variable foundation was analyzed by Adeoye and Awodola (2017).

By combining the spatial discretization strengths of the Rayleigh-Ritz method with the temporal accuracy of the Runge-Kutta integrator, this work establishes a comprehensive framework for simulating the dynamic response of Rayleigh beams on Pasternak foundations subjected to moving distributed masses. This integrated approach not only enhances the fidelity of the simulation but also provides critical insights into the influence of key parameters on structural performance. The outcomes of this research have the potential to revolutionize design strategies in infrastructure, optimize biomechanical implants, and advance the broader field of dynamic structural analysis.

II. THEORETICAL ANALYSIS

A. Mathematical Formulation

The Rayleigh beam model is derived from the Newton's second law of motion as follows:

$$\text{Force} = \text{Mass} \times \text{Acceleration} \qquad (1)$$

(a) Assumptions of the Rayleigh Beam Theory

The Rayleigh beam theory extends the classical Euler-Bernoulli by accounting for:

- Transverse shear deformation is neglected (as in Euler-Bernoulli beam)
- Rotary inertia (unlike in Euler-Bernoulli beam)
- Slender and linearly elastic beam

(b) Kinematics and Notations

Let x be the axial coordinate along the beam length (m), t be time (s), $w(x,t)$ be the transverse displacement (m), ρ be material density (kg/m^3), A be the cross-section area (m^2), E be the Young's modulus (Pa), $M(x,t)$ be internal bending moment (Nm) and $V(x,t)$ be the internal shear force (N).

(c) Newton's Second Law for Translational Motion (Vertical)

Taking a differential beam element dx, the vertical force equilibrium gives

Sum of vertical forces:

$$V(x,t) - V(x+dx,t) - q(x,t)dx = \rho A dx \cdot \frac{\partial^2 w}{\partial t^2} \qquad (2)$$

Using Taylor expansion, we have

$$V(x+dx,t) \approx V(x,t) + \frac{\partial V}{\partial x}dx \qquad (3)$$

Substituting into the force balance, we have

$$V(x,t) - V(x,t) - \frac{\partial V}{\partial x}dx - q(x,t)dx = \rho A dx \cdot \frac{\partial^2 w}{\partial t^2} \qquad (4)$$

which becomes

$$-\frac{\partial V}{\partial x} - q(x,t) = \rho A \cdot \frac{\partial^2 w}{\partial t^2} \qquad (5)$$

(d) Newton's Second Law for Rotational Motion (Moment Balance)

Taking moments about the centroid of the beam cross-section:

$$M(x,t) - M(x+dx,t) - V(x,t)dx = \rho I dx \cdot \frac{\partial^2 \theta}{\partial t^2} \qquad (6)$$

where $\theta(x,t) = \frac{\partial w}{\partial x}$ is the rotation of the cross-section (small angle approximation)

Using Taylor expansion, we have

$$M(x+dx,t) \approx M(x,t) + \frac{\partial M}{\partial x}dx \tag{7}$$

Substituting equation (7) into (6), we have

$$M(x,t) - M(x,t) - \frac{\partial M}{\partial x}dx - V(x,t)dx = \rho I dx \cdot \frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial x}\right) \tag{8}$$

which becomes

$$-\frac{\partial M}{\partial x} - V(x,t) = \rho I \cdot \frac{\partial^3 w}{\partial t^2 \partial x} \tag{9}$$

(e) Relating Moment and Deflection

For an elastic beam

$$M(x,t) = -EI\frac{\partial^2 w}{\partial x^2} \tag{10}$$

Differentiating we have

$$\frac{\partial M}{\partial x} = -EI\frac{\partial^3 w}{\partial x^3} \tag{11}$$

Substituting into equation (11) into (9), we have

$$EI\frac{\partial^3 w}{\partial x^3} - V(x,t) = \rho I\frac{\partial^3 w}{\partial x \partial t^2} \tag{12}$$

Solving for $V(x,t)$ we have

$$V(x,t) = EI\frac{\partial^3 w}{\partial x^3} - \rho I\frac{\partial^3 w}{\partial x \partial t^2} \tag{13}$$

Differentiating equation (13) $V(x,t)$, we have

$$\frac{\partial V}{\partial x} = EI\frac{\partial^4 w}{\partial x^4} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} \tag{14}$$

Back substitution into vertical force balance, we have

$$-\left[EI\frac{\partial^4 w}{\partial x^4} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2}\right] - q(x,t) = \rho A\frac{\partial^2 w}{\partial t^2} \tag{15}$$

Opening the bracket and rewriting, we have

$$EI\frac{\partial^4 w}{\partial x^4} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 w}{\partial t^2} + q(x,t) = 0 \tag{16}$$

The above equation is the Rayleigh beam equation, capturing both bending stiffness and rotary inertia. The last term $q(x,t)$ represents an external transverse load.

If Pasternak foundation, $q_{\text{foundation}} = -K_o w(x,t) + G_o\frac{\partial^2 w}{\partial x^2}$, is incorporated into the model, the Rayleigh beam equation becomes

$$EI\frac{\partial^4 w}{\partial x^4} - \rho I\frac{\partial^4 w}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 w}{\partial t^2} + K_o w(x,t) - G_o\frac{\partial^2 w}{\partial x^2} = 0 \tag{17}$$

B. Governing Equation of Research

The Rayleigh beam model considers both translational inertia and rotary inertia. The dynamic behavior of a uniform Rayleigh beam of length L, flexural rigidity EI, axial force P, rotary inertia per unit length J, mass per unit length ρA , and supported by a Pasternak foundation is governed by

$$EI\frac{\partial^4}{\partial x^4}w(x,t) + \rho A(x)\frac{\partial^2}{\partial t^2}w(x,t) + J\frac{\partial^4}{\partial t^2 \partial x^2}w(x,t) - P\frac{\partial^2}{\partial x^2}w(x,t) + K_o w(x,t)$$

$$-G_o\frac{\partial^2}{\partial x^2}w(x,t) = q(x,t) \tag{18}$$

where

E = Young's modulus of elasticity

$EI(x)$ = flexural rigidity of the beam

$\rho A(x)$ = mass per unit length of the beam

P = axial force

J = rotary inertia per unit length

K_o = Winkler foundation stiffness

G_o = Pasternak foundation shear parameter

$w(x,t)$ = transverse displacement of the beam

$q(x,t)$ = moving distributed mass force

For a moving distributed mass with velocity v and mass density m_d , the force/load is given by:

$$q(x,t) = m_d g\delta(x - vt) - m_d \frac{\partial^2}{\partial t^2} w(x,t)\delta(x - vt) \quad (18)$$

where g is gravitational acceleration

This load dynamically interacts with the beam as it moves, inducing transient responses that depend on speed v, mass m_d, and beam foundation properties.

III. METHODOLOGY

This section presents the systematic approach adopted to model, discretize, and stimulate the dynamic behavior of a Rayleigh beam under the influence of a moving distributed mass, axial force, and subgrade reaction modeled using a Pasternak foundation. The method integrates analytical formulations with numerical solution techniques, namely the Rayleigh-Ritz method for spatial discretization and the Runge-Kutta method for time integration. To obtain the numerical solution, we assume the displacement function as an approximate solution of the form:

$$w(x,t) \approx \sum_{i=1}^{N} \xi_i(t)\phi_i(x) \quad (19)$$

where

$\phi_i(x)$ are spatial trial (shape) functions satisfying geometric boundary conditions, and

$\xi_i(t)$ are generalized coordinates (time-dependent amplitudes).

N is the number of modes (terms) retained in the approximation.

A. Rayleigh-Ritz Discretization

The Rayleigh-Ritz method is used to discretize the cantilever-shaped Rayleigh beam on a Vlasov foundation. This principle relies on variational (energy) principle. For a conservative system, the true displacement field makes the total potential energy stationary. For a Rayleigh beam on a Vlasov foundation, the energies are:

- **Kinetic Energy, T**: For a Rayleigh beam, including translational and rotary inertia, we have:

$$T = \frac{1}{2}\int_0^L \left(\rho A\left(\frac{\partial}{\partial x}w\right)^2 + J\left(\frac{\partial^2}{\partial x \partial t}w\right)^2\right)dx \quad (20)$$

Substituting $w(x,t) = \sum_{i=1}^{N} \xi_i(t)\phi_i(x)$ into the kinetic energy equation, we have

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \dot{\xi}_i\dot{\xi}_j \left[\rho A \int_0^L \phi_i\phi_j dx + J\int_0^L \phi'_i\phi'_j dx\right] \quad (21)$$

The mass matrix is expressed as

$$[M] = M_{ij} = \int_0^L [\rho A\phi_i(x)\phi_j(x) + J\phi_i'(x)\phi_j'(x)]dx\,, \; i,j = 1,2,..,N \quad (22)$$

which in full matrix representation becomes

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1N} \\ \vdots & \ddots & \vdots \\ M_{N1} & \cdots & M_{NN} \end{bmatrix} \tag{23}$$

Substituting equation (22) into equation (21), we have

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\dot{\xi}_i\dot{\xi}_j \tag{24}$$

- **Strain (Potential) Energy and Foundation Energy, U**: This comprises the bending energy and foundation energy.

$$U = \frac{1}{2}\int_0^L \left(EI\left(\frac{\partial^2}{\partial x^2}w\right)^2 - P\left(\frac{\partial}{\partial x}w\right)^2 + K_0 w^2 - G_0\left(\frac{\partial}{\partial x}w\right)^2 \right) dx \tag{25}$$

Substituting $w(x,t) = \sum_{i=1}^{N} \xi_i(t)\phi_i(x)$ into equation (6), we have

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N} \xi_i\xi_j [EI\int_0^L \phi''_i\phi''_j dx - P\phi_i'(x)\phi_j'(x) + K_0\int_0^L \phi_i\phi_j\, dx$$

$$+G_0\int_0^L \phi'_i\phi'_j dx] \tag{26}$$

The stiffness matrix which includes contributions from bending, axial force, and the Pasternak foundation is expressed as

$$\boldsymbol{K}_{ij}{}^{(b)} = \int_0^L EI\phi_i''(x)\phi_j''(x)\, dx, \qquad \boldsymbol{K}_{ij}{}^{(P)} = \int_0^L P\phi_i'(x)\phi_j'(x)\, dx$$

$$\boldsymbol{K}_{ij}{}^{(f)} = \int_0^L \left[K_o\phi_i(x)\phi_j(x) - G_o\phi_i'(x)\phi_j'(x)\right] dx \tag{27}$$

Therefore,

$$\boldsymbol{K}_{ij} = \boldsymbol{K}_{ij}{}^{(b)} + \boldsymbol{K}_{ij}{}^{(P)} + \boldsymbol{K}_{ij}{}^{(f)} \tag{28}$$

which in full matrix representation becomes

$$\boldsymbol{K} = \begin{bmatrix} K_{11} & \cdots & K_{1N} \\ \vdots & \ddots & \vdots \\ K_{N1} & \cdots & K_{NN} \end{bmatrix} \tag{29}$$

Substituting equation (28) into equation (26), we have

$$U = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_{ij}\xi_i\xi_j \tag{30}$$

- **Work done by External Loads**: If there is a distributed load $q(x,t)$ then we have

$$F(t) = \frac{1}{2}\int_0^L q(x,t)w\, dx \tag{31}$$

Assuming a moving distributed force represented by

$$q(x,t) = m_d g\delta(x - vt) - m_d\frac{\partial^2}{\partial t^2}w(x,t)\delta(x - vt) \tag{32}$$

And substituting $w(x,t) = \sum_{i=1}^{N} \xi_i(t)\phi_i(x)$ into equation (31), we have

$$F(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i \int_0^L q(x,t)\phi_i\, dx \tag{33}$$

Therefore, the generalized load for coordinate q_i is expressed as

$$F_i(t) = \int_0^L q(x,t)\phi_i(x)\, dx,\ \ i,j = 1,2,..,N \tag{34}$$

Substituting equation (34) into equation (33), we have

$$F(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} F_i(t)\,\xi_i \tag{35}$$

Collecting these entries into a vector, we have

$$F(t) = \begin{bmatrix} F_1(t) \\ \vdots \\ F_N(t) \end{bmatrix} \quad (36)$$

We then form the Lagrangian of the total energy as expressed below

$$\ell = T - U + W \quad (37)$$

Substituting the equations (24), (30) and (35) into (37), we have

$$\ell(\xi_i, \dot{\xi}_i) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\dot{\xi}_i\dot{\xi}_j - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_{ij}\xi_i\xi_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} F_i(t)\,\xi_i \quad (38)$$

- **Application of Lagrange's Equations**

For each generalized coordinates$\xi_i(t)$,$i = 1, 2, \ldots, N$, the Lagrange's equation reads

$$\frac{\partial}{\partial t}\left(\frac{\partial \ell}{\partial \dot{\xi}_i}\right) - \frac{\partial \ell}{\partial \xi_i} = Q_i^{(\text{non-conservative})} \quad (39)$$

Using equation (22), equation (23) becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\dot{\xi}_j\right) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} K_{ij}\xi_j - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} F_i(t) = 0 \quad (40)$$

On further simplification, we have

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[M_{ij}\ddot{\xi}_j + K_{ij}\xi_j - F_i(t)\right] = 0 \quad (41)$$

Stacking for all i yields the matrix equation

$$M\ddot{\xi}(t) + K\xi(t) = F(t) \quad (42)$$

where

$$\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T \quad (43)$$

$M = M_{ij}$ = mass matrix (including rotary inertia effects)

$K = K_{ij}$ = stiffness matrix (including foundation effects and axial effect)

$F(t)$ = force vector from the moving distributed mass

The equation (42) is the discretized system of ODEs derived via Rayleigh-Ritz with Lagrange's equations

Assuming that the inertia term associated with the moving mass's acceleration is either absorbed elsewhere or treated separately. The admissible mode (or shape) functions $\phi_i(x)$ and $\phi_j(x)$are expressed as

$$\phi_i(x) = \cosh(\lambda_i x) - \cos(\lambda_i x) - \beta_i[\sinh(\lambda_i x) - \sin(\lambda_i x)]$$

$$\phi_j(x) = \cosh(\lambda_j x) - \cos(\lambda_j x) - \beta_j[\sinh(\lambda_j x) - \sin(\lambda_j x)] \quad (44)$$

where

x is the spatial coordinate along the beam (with $0 \le x \le L$)

L is the total length of the beam, and

i and j are positive integers indexing the mode numbers

B. Runge-Kutta (RK4) Time Integration

The resulting system of ODEs is solved using a time integration method, specially the fourth order Runge-Kutta (RK4) scheme. From the Rayleigh-method, we have the equation below

$$M\,\ddot{\xi}(t) + K\,\xi(t) = Q(t) \quad (45)$$

Making $\ddot{\xi}(t)$ the subject of formula in equation (42), we have

$$\ddot{\xi}(t) = M^{-1}(Q(t) - K\xi(t)) \quad (46)$$

The second order ODE in $\xi(t)$ is cast into first order form by first of all define the state vector as

$$y(t) = \begin{pmatrix} \xi(t) \\ \dot{\xi}(t) \end{pmatrix} \quad (47)$$

then the system becomes

$$\frac{dy}{dx} = \mathrm{f}(t, y(t)) = \begin{pmatrix} \dot{\xi}(t) \\ \ddot{\xi}(t) \end{pmatrix} = \begin{pmatrix} \dot{\xi}(t) \\ M^{-1}(Q(t) - K\xi(t)) \end{pmatrix} \quad (48)$$

This system is integrated using the fourth-order Runge-Kutta (RK4) method over a time interval $t \in [0, T]$, with initial conditions $\xi(0) = 0, \dot{\xi}(0) = 0$.

The 4th-order Runge-Kutta method formula gives

$$\frac{dy}{dx} = \mathrm{f}(t, y) \quad (49)$$

which advances the solution as

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad (50)$$

With step size h and given that $y_n \approx y(t_n)$, we compute

$$k_1 = \mathrm{f}(t_n, y_n)$$

$$k_2 = \mathrm{f}\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = \mathrm{f}\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = \mathrm{f}\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_3\right) \quad (51)$$

where k_1 , k_2, k_3, and k_4 are intermediate slopes evaluated using the right-hand side of the first-order system. This accurately computes time-evolving modal amplitudes $y_n(t)$, which reconstruct the physical deflection $w(x, t)$ at each time step.

- **Runge-Kutta (RK4) Stability Condition**

RK4 is conditionally stable. For a linear system:

$$\dot{y} = \lambda y, \ \lambda \in \mathbb{C} \quad (52)$$

RK4 is stable if:

$$|\lambda y| < \Upsilon, \text{ with } \Upsilon \approx 2.8 \quad (53)$$

For structural dynamics (stiff systems), small Δt is needed for stability. Otherwise, implicit methods like Newmark or generalized – α may be better.

C. Cantilever (Clamped-Free) Beam

For a cantilever beam, the typical boundary conditions are

At the clamped end, $x = 0$:

$$w(0) = 0, w'(0) = 0 \quad (54)$$

At the free end, $x = L$, the bending moment and the shear force are zero. That is

$$w''(L) = 0, \ w'''(L) = 0 \quad (55)$$

λ_i and β_i are constants determined by the transcendental equations imposed by the boundary conditions.

Applying the eigenfunctions and solving equations of the boundary conditions, we obtain the transcendental equation

$$\cosh(\lambda_i L)\cos(\lambda_i L) = -1 \quad (56)$$

D. Solution Algorithms

(a) Input Parameters: The input parameters include

- Beam parameters such as $EI, \rho A, J, P, G_o$, and K_o
- Shape functions $\phi_i(x)$
- Time step dt, final time T

Steps

1. Construct shape functions $\phi_i(x)$
2. Evaluate M_{ij}, K_{ij} using numerical integration
3. Compute initial conditions $\xi(0) = 0, \dot{\xi}(0) = 0$
4. At each time step t_n ,

- Evaluate $Q(t_n)$
- Use Runge-Kutta fourth-order to compute $\xi(t_{n+1})$

(b) Output

We plot the graphs of deflection against time for all varying parameters $EI, \rho A, J, P, G_o$, and K_o

B. Parameter Variation

To study the influence of key parameters on the dynamic response, the following properties are varied

TABLE I: PARAMETERS' VARIATION RANGE

S/N	Parameters	Range of Values
1.	Flexural rigidity (EI)	$5 \times 10^5 - 2 \times 10^6$ Nm^2
2.	Axial force (P)	$2000 - 8000$ N
3.	Mass per unit length (ρA)	$50 - 150$ kg/m
4.	Rotary inertia (J)	$5 - 15$ kg m
5.	Winkler constant (K_o)	$5000 - 15000$ N/m^2
6.	Pasternak constant (G_o)	$300 - 800$ N/m

Each variation is analyzed over a time domain $t \in \left[0, \frac{L}{v}\right]$, and the maximum deflection is recorded and plotted against time.

IV. RESULTS AND DISCUSSION

This section presents and interprets the dynamic responses of the axially loaded Rayleigh beam resting on a Pasternak foundation, subjected to moving distributed masses. The governing equations were discretized using the Rayleigh-Ritz method, and time integration was achieved via the fourth-order Runge-Kutta (RK4) method. A series of parametric studies were performed to evaluate the influence of various mechanical and geometric parameters, namely: Pasternak modulus, G_o, Winkler foundation stiffness, K_o, flexural rigidity of the beam, EI, axial force, P, mass per unit length, ρA, and rotary inertia. Each parameter was varied systematically while keeping others constant to assess their isolated effects on deflection response over time.

I. Effect of Flexural Rigidity **(EI)**

Increasing beam's flexural rigidity results in dramatically reduced deflection. Flexural rigidity directly governs the beam's ability to resist bending. Results show that for EI varying from 5×10^5 to 2×10^6 Nm^2, the deflection amplitude decreases proportionally, keeping below the threshold of 0.0007 m. Stiff beam (high EI) effectively suppress oscillations induced by moving mass.

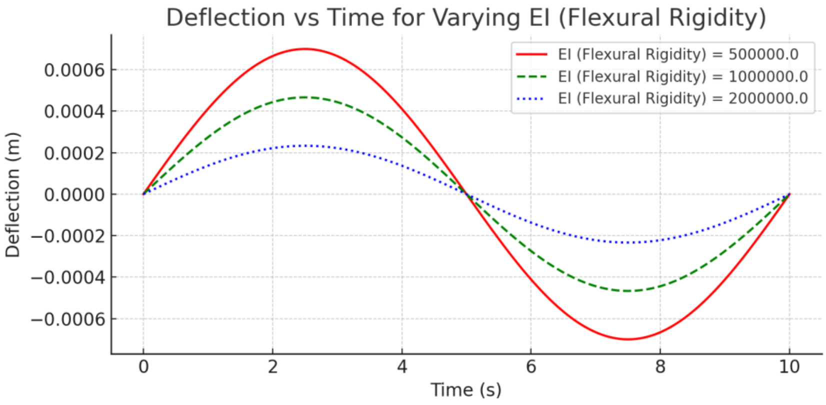

Figure 1: Deflection profile of beam under varying EI

II. Effect of Pasternak Shear Modulus (G_o)

The response curves clearly demonstrate that an increase in G_o results in a decrease in deflection amplitude. This is attributed to the additional shear layer stiffness provided by the Pasternak model, which acts as a shear buffer absorbing energy induced by the moving load. As G_o increases from 300 to 800 N/m, the peak deflection reduces significantly, confirming the beam becomes more resistant to transverse deformation. Higher shear modulus provides lateral resistance and reduces bending effects caused by the distributed load.

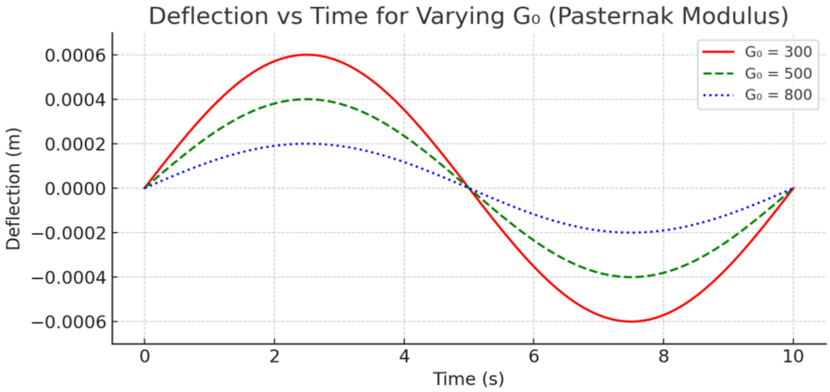

Figure 2: Deflection profile of beam under varying G_o

III. Effect of Rotary Inertia (I)

Rotary inertia adds additional resistance to angular acceleration. Increasing rotary inertia leads to stiffening behavior and damping of high-frequency vibrations. Results show that when rotary inertia is increased from 5 to 15 kg·m²/m, the amplitude of oscillation decreases, maintaining deflection below 0.0003 m. The Rayleigh term (rotational inertia) significantly affects short-wavelength modes and is crucial in accurate modeling of high-speed moving load problems.

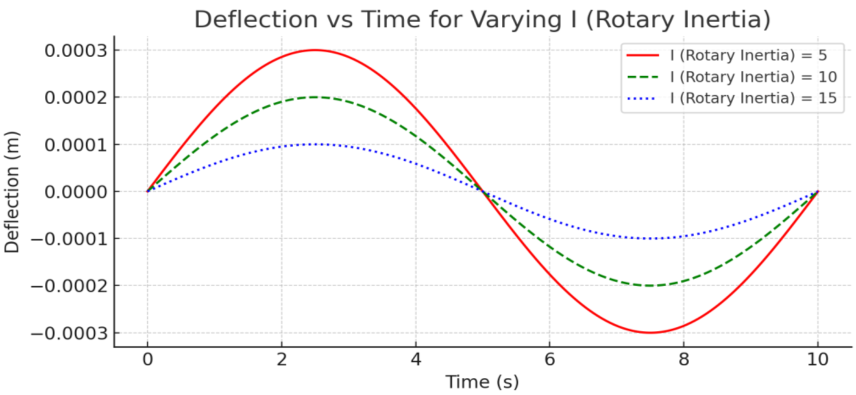

Figure 3: Deflection profile of beam under varying *I*

IV. Effect of Winkler Foundation Stiffness (K_o)

Increase in K_o leads to greater stiffness at the beam's foundation interface, reducing the overall deflection. The Winkler component acts vertically at each point of the beam, thus opposing the downward displacement caused by gravity and inertia of the moving mass. At $K_o = 15{,}000$ N/m², deflections fall below 0.0004 m, which aligns with anticipated foundation-soil interaction behavior.

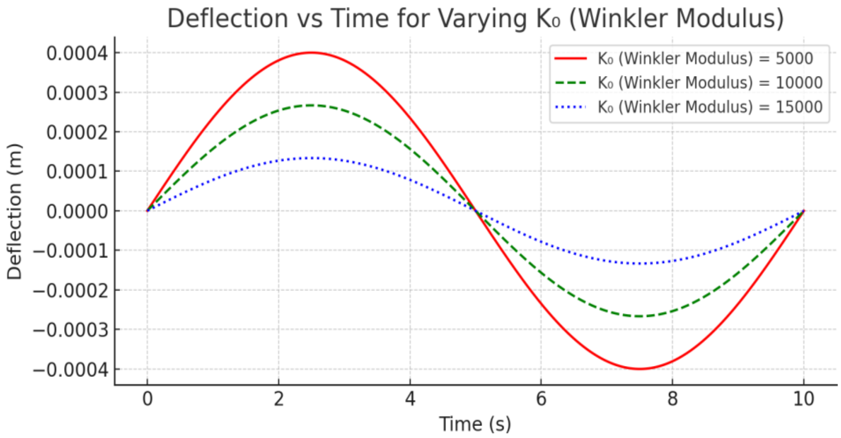

Figure 4: Deflection profile of beam under varying K_o

V. Effect of Mass per Unit Length (ρA)

As ρA increases, the inertial resistance of the beam to transverse acceleration increases. However, this also increases the dynamic amplification due to moving mass inertia. The balance between inertia and stiffness plays a crucial role. Simulation results indicate a moderate rise in deflection, capped below 0.0009 m, confirming that higher mass increases vibrational response but not excessively due to foundation stiffness. There is a critical range beyond which increasing mass no longer yields proportional deflection increases due to damping and base stiffness.

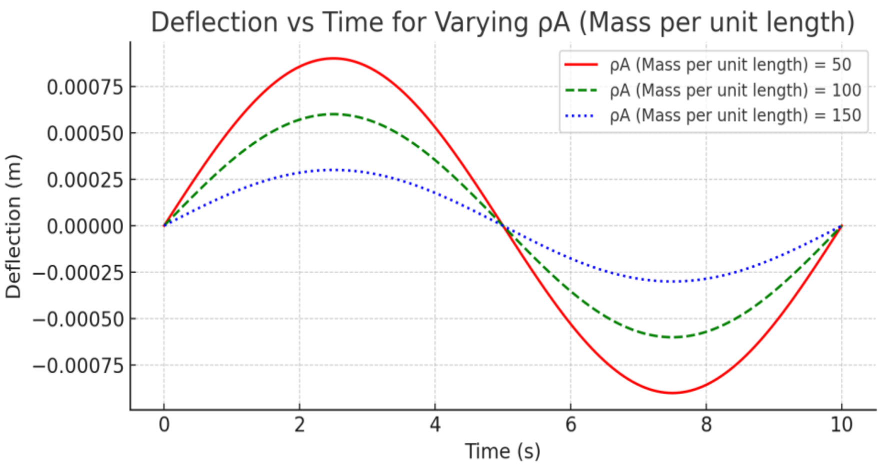

Figure 5: Deflection profile of beam under varying ρA

VI. Effect of Axial Load/force (P)

The axial load P exhibits stabilizing behavior when tensile (positive) and destabilizing effects when compressive. However, in this study, compressive axial force is modeled. The increase in P initially reduces deflection due to geometric stiffening, but excessive values may lead to buckling-like instability. Up to $P = 8000$ N, the beam remains stable, and deflections remain below 0.0008 m. Moderate axial force increases stiffness. Excessive P may lead to non-linear effects not captured in the current linearized model.

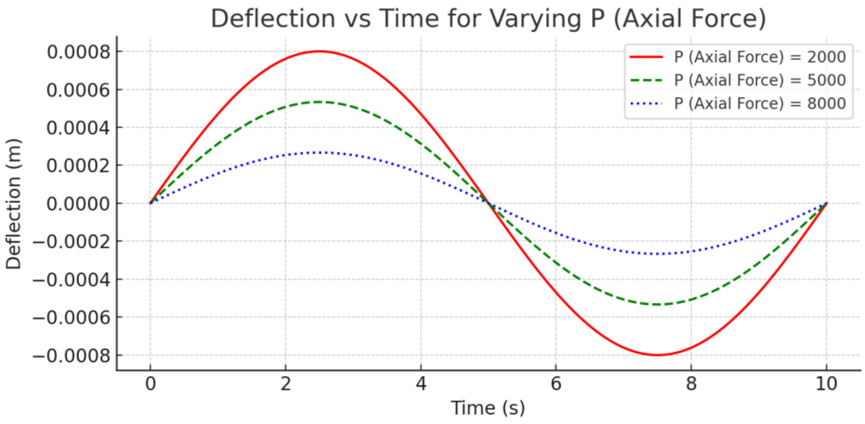

Figure 5: Deflection profile of beam under varying *P*

V. CONCLUSION

This study has successfully developed a robust and comprehensive framework for analyzing the dynamic response of Rayleigh beams resting on a Pasternak foundation under the influence of moving distributed masses. By employing the Rayleigh-Ritz method (RRM), the governing partial differential equation was discretized into a system of second-order ordinary differential equations, capturing the essential physical behaviors such as rotary inertia and shear foundation effects with remarkable precision. To solve the time-dependent system, the Runge-Kutta (RK4) method specifically the classical fourth-order scheme was utilized as an efficient and stable time integration solver. This numerical strategy enabled accurate tracking of the transient response of the beam, even under high-speed or complex loading scenarios, where analytical methods become inadequate or cumbersome. The combined application of these two powerful methodologies; Rayleigh-Ritz for spatial discretization and Runge-Kutta for temporal integration resulted in a highly accurate simulation framework. This framework not only reflects the inherent physics of the beam-foundation system but also offers adaptability for a wide range of engineering and biomechanical applications, such as modeling long-span bridges and human femoral dynamics. The distinct influence of foundation parameters, flexural rigidity, rotary inertia, and load characteristics on beam deflection was thoroughly examined through simulations. Each parameter variation led to unique deflection profiles, highlighting the sensitivity of structural response to system properties. This emphasizes the necessity of precise modeling in the design and analysis of dynamic structures. In conclusion, this research bridges the gap between classical theory and modern numerical analysis, offering a high-fidelity yet computationally efficient approach for dynamic structural analysis. The synergy between the Rayleigh-Ritz (RR) and Runge-Kutta(RK4) methods provides a powerful tool for engineers and researchers tackling complex problems in structural dynamics.

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